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A nonlocal theory of sediment transport on hillslopes

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[1] Hillslopes are typically shaped by varied processes which have a wide range of event-based downslope transport distances, some of the order of the hillslope length itself. We hypothesize that this can lead to a heavy-tailed distribution of displacement lengths for sediment particles. Here, we propose that such a behavior calls for a nonlocal computation of the sediment flux, where the sediment flux at a point is not strictly a function (linear or nonlinear) of the gradient at that point only but is an integral flux taking into account the upslope topography (convolution Fickian flux). We encapsulate this nonlocal behavior in a simple fractional diffusive model which involves fractional derivatives, with the order of differentiation ($1 < \alpha \leq 2$) dictating the degree of nonlocality ($\alpha = 2$ corresponds to linear diffusion and strictly local dependence on slope). The model predicts an equilibrium hillslope profile which is parabolic close to the ridgetop and transits, at a short downslope distance, to a power law with an exponent equal to the parameter α of the fractional transport model. Hillslope profiles reported in previously studied sites support this prediction. Furthermore, we show that the nonlocal transport model gives rise to a nonlinear dependency on local slope and that variable upslope topography leads to widely varying rates of sediment flux for a given local hillslope gradient. Both of these results are consistent with available field data and suggest that nonlinearity in hillslope flux relationships may arise in part from nonlocal transport effects in which displacement lengths increase with hillslope gradient. The proposed hypothesis of nonlocal transport implies that field studies and models of sediment fluxes should consider the size and displacement lengths of disturbance events that mobilize hillslope colluvium.

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1. Introduction

[2] In absence of overland flow-driven or wind-driven transport, the movement of soil on landscapes requires some kind of disturbance (Figure 1). This disturbance arises in many ways leading to a wide range of length scales of displacement. In clay-rich soils mantling sloping landscapes, periodic wetting of the ground may cause swelling and downslope flow, but even as the soils remain wet, progressively increasing grain resistance may halt motion. Drying and cracking then resets the contacts and allows another period of flow in the next wet season [*Fleming and Johnson*, 1975]. This cycle operates over some length scale of displacement. Simple wetting expansion and drying collapse through a season can incrementally shift near surface soils short distances downslope [e.g., *Kirkby*, 1967]. Seasonal cycles of movement by ice-driven processes shift soils and during spring melt can give way as continuously moving solifluction lobes which may carry soil a considerable distance even on gentle slopes [e.g., *Washburn*, 1973]. Biota work the soil at a wide range of scales, leading to dilation and displacement downslope. Insects and worms may cause minor local displacement but through their persistent and pervasive activity cause significant movement [e.g., *Darwin*, 1881]. Burrowing animals can make an extensive network of tunnels and push piles of dirt meters downslope. The collapse of large trees may rotate and expose their root system and displace clumps of soil meters downslope [e.g., *Norman et al.*, 1995; *Gabet et al.*, 2003]. The exposed, locally steep, tree throw mound and the smaller annual burrow mounds are sites of accelerated rain splash, raveling and fine scale biotic disturbance. In effect, the biotic roughening of the ground surface by the local mound formation leads to accelerated soil movement. On sufficiently steep granular soils, fire may suddenly remove particles stored behind fallen woody debris and unleash particles to ravel downslope [e.g., *Roering and Gerber*, 2005], sometimes tens of meters. Shallow landslides may also initiate, mobilize, and redeposit on hillslopes. Soil movement, then, arises through the sum of stochastic processes, influenced by seasonal and biotic cycles, the integral of which is a net flux of soil which tends to increase with

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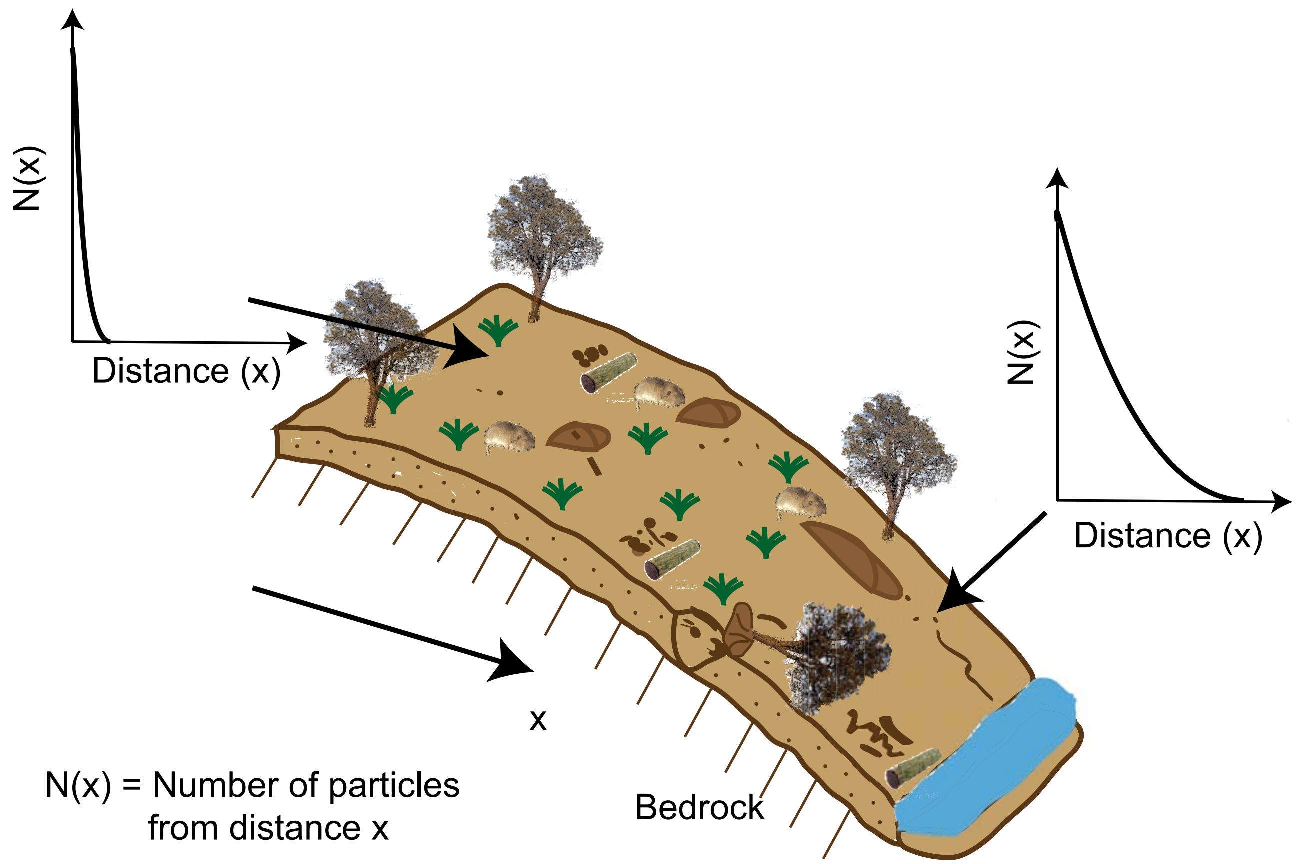

Figure 1. Cartoon illustrating processes such as gopher mounds, tree throws, and wood blockage, which contribute to sediment transport on a hillslope. Owing to the varied range of length scales of transport of these processes, the number of sediment particles arriving at a given location downslope is influenced by a region of upslope topography. This can be treated using the notion of a nonlocal flux (equation (8)) which is computed by a weighted average of upslope contributions.

increasing hillslope gradient. The individual particle step lengths resulting from disturbances will vary greatly.

[3] On gentle hillslopes there is field evidence [e.g., *McKean et al.*, 1993] that the mean soil transport varies linearly with local gradient. On steeper slopes, however, theory and limited observations suggest that transport increases nonlinearly with slope [e.g., *Roering et al.*, 1999]. Increasing field and theoretical evidence indicates that flux also depends on active transport depth [*Heimsath et al.*, 1999; *Roering*, 2008, *Furbish et al.*, 2009]. In particular, *Furbish et al.* [2009] show that a diffusivity-like coefficient which takes into account the local slope depth product produces a sediment flux which varies linearly with local gradient. Both linear and nonlinear flux laws assume that transport depends on some "local" slope, although we lack theory for what sets the length scale over which that slope should be determined. The disturbance by biota creates an irregular ground surface, with locally steep piles of loose soil that diffuse downslope across the mean slope (Figure 1). Hence, the slope at any point may not represent the actively contributing slope-driving processes, and cannot account for travel distances resulting from disturbances. If we could monitor every particle on a hillslope where these disturbance-driven processes (often placed together under the term "creep") occur, it is possible that long transport events occur with a finite, nonvanishing, nonexponentially decaying probability such that the pdf of transport distances is heavy tailed [e.g., *Tucker and Bradley*, 2010]. This conception of soil transport may not be well represented by a transport expression that relates flux to a "local" slope. Moreover, the possibility of heavy-tailed particle travel histories makes selecting a meaningful mean slope for the application of such local laws problematic. To date, empirical fitting procedures (reducing variance by increasing the length scale of averaging while trying to maintain local profile curvature) have been used for the estimation of the local slope; common methods include polynomial fitting and Gaussian filtering [e.g., *Roering et al.*, 1999; *Lashermes et al.*, 2007].

[4] Here we propose an alternative formulation of sediment transport on hillslopes which relies on the notion of nonlocal computation of sediment flux, reflecting the fact that mass flux at a point on the hillslope is being influenced by disturbances well upslope and not simply linked to local slope (and soil depth). Our analysis may also explain the variance in flux rate for a given local slope observed in some studies. Our theory, although not derived from physical considerations (e.g., involving balances of forces and resistances), presents a general mathematical framework within which the upslope influences to the sediment flux at a given point can

be cast into a continuum constitutive law for sediment transport. Specifically, we propose a nonlocal formulation of transport laws which relies on an integral (non-Fickian) flux computation which explicitly takes into account the upslope topography from any point of interest. The proposed nonlocal transport model includes linear diffusive transport as a special case.

[5] The paper is structured as follows. In section 2, we formulate the nonlocal constitutive law for sediment transport on hillslopes and in section 3 we derive its steady state equilibrium profile under appropriate boundary conditions. In section 4 we interpret observed hillslope profiles in the Oregon Coast Range, in the Appalachians of Maryland and Virginia, and east of San Francisco (California) within the nonlocal transport formulation. In section 5 we compare the linear, nonlinear and nonlocal transport models in several ways. The most important result is that the linear nonlocal model gives rise to a nonlinear relationship between sediment flux and local slope, akin to that observed on steep slopes. In section 6 we demonstrate that applying the nonlocal flux model to an ensemble of hillslope profiles produces significant variability of sediment flux for a given value of local slope as a result of variations in upslope topography. In section 7, we discuss the relationship between the shape of the probability density function of the sediment displacement lengths (which dictate the microscopic behavior of the transport process but which are typically not measured) and the parameter α of the nonlocal transport model (which describes the macroscopic properties of the transport). In section 8 we present some preliminary thoughts as to the ability of the nonlocal transport formulations to circumvent the scale dependence of sediment flux computed using local, nonlinear models. We conclude that our model shows the possibility that nonlocal sediment transport processes may be important on hillslopes and warrant more consideration both in field studies and theoretically. Our model anticipates more process-based considerations that would account mechanistically for biotic disturbance and it suggests that models for transport and weathering of colluvial soils and geochronological analysis of particles on steep hillslopes should consider the possible effects of nonlocal transport.

2. A Nonlocal Constitutive Law for Hillslope Sediment Transport: Convolution Fickian Flux

[6] The simplest sediment flux law, proposed by *Culling* [1960] in analogy to Fick's law of diffusion, expresses sediment flux as proportional to the topographic gradient:

$$q_s(x) = -K\nabla h(x) \tag{1}$$

where $q_s(x)$ is sediment flux (volume per unit time per unit width: $L^3/L/T$) at location x (where x is distance from the ridgetop), K is the diffusivity coefficient (L^2/T), and $h(x)$ is the surface elevation with respect to a datum. It is easy to show [e.g., *Howard*, 1994] that substituting (1) in the continuity (Exner) equation:

$$\rho_r \frac{\partial h}{\partial t} = \rho_r U - \rho_s \nabla \cdot q_s \tag{2}$$

where ρ_s and ρ_r are the bulk densities of sediment and rock, respectively, and U is the rock uplift rate results in the linear diffusion equation:

$$\frac{\partial h}{\partial t} = U + K\nabla^2 h \tag{3}$$

where we have assumed for simplicity that the bulk densities of rock and sediment are the same (which is almost never the case) and have ignored chemical erosion. (Note that equation (3) can also be derived using a moving coordinate system of erosion driven by diffusive transport in which the uplift term enters as a lower boundary condition.) If the rate of surface erosion is approximately balanced by the rock uplift, i.e., dynamic equilibrium [*Gilbert*, 1909; *Hack*, 1960], then $\partial h/\partial t \approx 0$ and the steady state 1D case can be written as

$$\frac{\partial h}{\partial t} = 0 \Leftrightarrow \frac{d^2 h}{dx^2} = -\frac{U}{K} \tag{4}$$

Integrating twice and imposing the boundary conditions

$$h(0) = H_{top} = \frac{U}{2K}L^2 \qquad \left.\frac{dh}{dx}\right|_{x=0} = 0 \tag{5}$$

such that $h(L) = 0$ (river edge), the solution is given by

$$h(x) = H_{top} - \frac{U}{2K}x^2 \tag{6}$$

for $0 \leq x \leq L$ [e.g., *Koons*, 1989]. Furthermore, the properties of the equilibrium hillslope profiles predicted by linear diffusion are (1) linear increase of local slope with downslope distance and (2) constant curvature along the hillslope profile.

[7] The underlying assumption of a classical diffusion equation is that the step lengths of sediment particles, defined as the distances traveled by the particles once entrained until they are deposited again on the surface, have a thin-tailed (e.g., exponential or Gaussian) distribution [e.g., *Ganti et al.*, 2010; *Schumer et al.*, 2009]. However, for the reasons discussed in the introduction, the distribution of step lengths of sediment particles may be heavy tailed; that is, they have a small but significant chance of traveling a large distance downslope. In such cases, the sediment flux at a point x has a significant contribution from a large upslope distance and thus a local computation of flux, such as that of equation (1), is no longer appropriate. Recently, a particle-based model for sediment transport on hillslopes was developed based on a plausible set of rules capturing disturbance-driven transport processes and it was shown that a heavy-tailed step length distribution can emerge due to the interactions between these disturbances and microtopography [*Tucker and Bradley*, 2010]. Here, we develop a continuum constitutive model for such a behavior. Specifically, we propose a notion of nonlocal sediment flux which takes into account the heavy tails in step lengths of sediment particles by expressing the sediment flux at a given

point as a weighted average of the upslope topographic attributes:

$$q_s^*(x) = -K^* \int_0^x g(l) \nabla h(x-l) dl \tag{7}$$

where $q_s^*(x)$ is sediment flux (volume per unit time per unit width: $L^3/L/T$) at location x (where x is distance from the ridgetop), K^* is the diffusivity coefficient, $h(x)$ is the topographic elevation at location x, and $g(l)$ is a kernel performing a weighted average of local gradients upslope of the point of interest x as they contribute to the sediment flux at the point x (Figure 1). This is a special case of the more general convolution Fickian flux laws [*Cushman*, 1991, 1997]. It has been shown [*Cushman and Ginn*, 2000] that when the weighting function $g(l)$ has no characteristic length scale, i.e., when $g(l)$ decays as a power law with the lag l, $g(l) \sim l^{1-\alpha}$, (7) takes the form of a fractional derivative:

$$q_s^*(x) = -K^* \nabla^{\alpha-1} h(x) \tag{8}$$

where $\alpha \in (1,2)$. Substituting (8) in the continuity equation (2) and making the assumption that bulk densities of rock and sediment are equal, leads to a fractional diffusion equation:

$$\frac{\partial h}{\partial t} = U + K^* \nabla^\alpha h \tag{9}$$

[8] The order of differentiation, α, directly relates to the heaviness of the distribution of step lengths [*Meerschaert et al.*, 1999, 2001; *Schumer et al.*, 2001, 2009] and $1 < \alpha < 2$ implies a distribution of step lengths with a finite population mean but infinite population variance (sample variance that diverges unstably as the number of samples increases) [*Lamperti*, 1962], resulting in an accelerated diffusion (superdiffusion). It is noted that for $\alpha = 2$, (8) becomes the standard Fickian flux (1), and (9) collapses to the linear diffusion equation (3).

[9] The concept of nonlocal transport, implemented via fractional derivatives or Continuous Time Randon Walk (CTRW) models, has been extensively used in other fields of study, such as subsurface transport [e.g., *Benson et al.*, 2000a; *Berkowitz et al.*, 2002], transport of pollutants in rivers [*Deng et al.*, 2005, 2006], hydrodynamics [e.g., *Metzler and Compte*, 2000], statistical mechanics [e.g., *Bouchaud and Georges*, 1990; *Pekalski and Sznajd-Weron*, 1999; *Shlesinger et al.*, 1995], molecular biology [e.g., *Campos et al.*, 2006] and turbulence [e.g., *Biler et al.*, 1998; *Woyczynski*, 1998]. Recently, it has been used in geomorphology to encapsulate the nonlocality of bed sediment transport along bedrock channels [*Stark et al.*, 2009] and to model the anomalous diffusion of tracer particles in gravel streams and sand bed rivers [*Ganti et al.* 2010; *Bradley et al.*, 2010]. A review of the application of partial fractional differential equations to the transport of solutes and sediment can be found in the work of *Schumer et al.* [2009].

3. Equilibrium Hillslope Profiles for Nonlocal Transport

[10] In order to derive the equilibrium hillslope profile for the fractional diffusion equation (9) we note that under dynamic equilibrium, the steady state 1-D equation can be written as

$$\frac{\partial h}{\partial t} = 0 \Leftrightarrow \frac{d^\alpha h}{dx^\alpha} = -\frac{U}{K^*} \tag{10}$$

[11] The two most commonly used definitions of a fractional derivative are the Riemann-Liouville and the Caputo forms [*Miller and Ross*, 1993]. These forms differ from each other in that the Riemann-Liouville definition expresses the fractional derivative as an integer order differential of a fractional integral (equation (11a)), whereas the Caputo definition expresses the fractional derivative as a fractional integral of an integer order derivative (equation (11b)):

$$\frac{d^\alpha h}{dx^\alpha} = \frac{d^n}{dx^n}\left(I_x^{n-\alpha} h(x)\right) \tag{11a}$$

$$\frac{d^\alpha h}{dx^\alpha} = I_x^{n-\alpha}\left(\frac{d^n h(x)}{dx^n}\right) \tag{11b}$$

where n is an integer such that $n - 1 < \alpha < n$ and $I_x^{n-\alpha}(\cdot)$ is a fractional integration operator of order $n - \alpha$. This distinction is important in the case of boundary-valued and initial-valued problems as the Riemann-Liouville definition requires the calculation of the derivatives of the fractional integrals of the function at the initial value, whereas the Caputo definition only requires the calculation of initial values of the function and its integer derivatives (see *Voller and Paola* [2010] for a detailed discussion). It is further worth noting that the Caputo fractional derivative (equation (11b)) of a constant is zero, and in this form a fractional integral and a fractional derivative are commutative, whereas the Riemann-Liouville fractional derivative (equation (11a)) of a constant is a power law. Specifically, the α-order fractional integral of a constant c is a power function:

$$I_x^\alpha\{c\} = \frac{c}{\Gamma(1+\alpha)} x^\alpha \tag{12}$$

where $I_x^\alpha\{\cdot\}$ is the fractional integral operator of order α, c is a constant and $\Gamma(\cdot)$ is the gamma function [*Oldham and Spanier*, 1974]. Implementation of the fractional derivative on a finite domain $0 \le x \le L$ with boundary conditions, requires defining the functional value $h(x)$ beyond the left boundary that is for $x < 0$. In a boundary-valued problem, the Caputo form of the fractional derivative assigns the values of the function (in this case $h(x)$) beyond the boundary to be equal to the value of the function at the boundary, i.e., it inherently assumes that $h(-\infty)$ up to $h(0)$ are assigned the value of $h(0) = H_{top}$. This, however, is physically unreasonable as no sediment is supplied at the ridge from any point beyond the ridge. In order to circumvent this issue we numerically evaluate the steady state equilibrium hillslope profiles predicted by equation (10).

[12] A fractional derivative can be discretized using the one-shift Grünwald expansion [*Meerschaert and Tadjeran*, 2004]:

$$\frac{d^\alpha h(x)}{dx^\alpha} \approx \frac{1}{\Delta x^\alpha} \sum_{k=0}^{N} g_k h(x - k\Delta x + \Delta x) \tag{13}$$

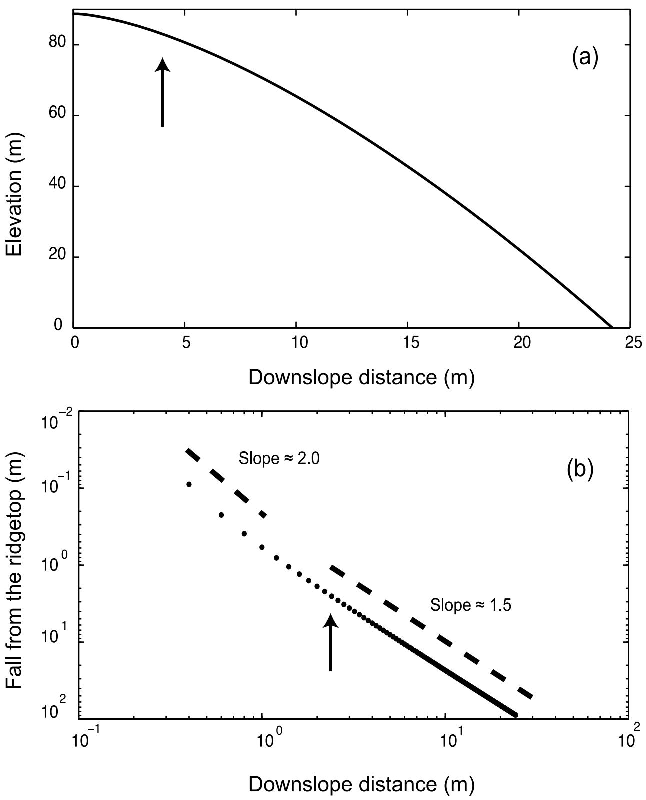

Figure 2. Steady state hillslope equilibrium profile predicted from fractional diffusive transport (equation (9)) with α = 1.5 and boundary conditions of zero slope at the ridge and zero elevation at the most downslope point. The parameter of the model K^* was chosen to be 1.0 $m^{1.5}$/yr and the rock uplift rate was set to unity [m/yr] (Note that a different value of rock uplift rate would not change the shape of the profile but only its absolute elevation would differ). (a) Profile shape and (b) log-log plot of vertical drop from the ridge top versus downslope distance. Notice the transition to a power law profile with exponent α = 1.5 at a distance of approximately 3 m from the ridgetop (arrow).

where g_k are the one-shift Grünwald weights, Δx is the spatial step size in the numerical implementation, N is the number of node points upslope of the given point and $1 < \alpha \leq 2$ is the order of differentiation. The Grünwald weights are given by the following expression [*Grünwald*, 1867; *Meerschaert and Tadjeran*, 2004]:

$$g_k = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)} \tag{14}$$

Imposing the boundary conditions

$$\begin{aligned} h(0) = H_{top} &= \frac{U}{\Gamma(1+\alpha)K^*}L^\alpha \\ \left.\frac{dh}{dx}\right|_{x=0} &= 0 \end{aligned} \tag{15}$$

such that $h(L) = 0$ at the river edge, and imposing an additional condition that $h(x) = 0$ for $x < 0$ (since there is no sediment supply to the domain from any point beyond the ridge), one can solve numerically for the steady state equilibrium hillslope profiles predicted by equation (10). Figure 2a shows the hillslope equilibrium profile for fractional transport with degree of nonlocality α = 1.5. It is noted that the hillslope profile is parabolic close to the ridge and transitions to a power law with an exponent of α.

[13] It is worth noting that under the Caputo form of the fractional derivative (which assumes that the values of $h(x) = H_{top}$ for $x < 0$), equation (10) can be solved analytically. The analytical solution of equation (10) with the boundary conditions (15) and $h(x) = H_{top}$ for $x < 0$ is given as

$$h(x) = H_{top} - \frac{U}{\Gamma(1+\alpha)K^*}x^\alpha \tag{16}$$

where x is the horizontal distance from the ridgetop, and H_{top} is the elevation of the ridgetop. As shown in Figure 2b, this solution is reached in the numerically evaluated profile (which assumes $h(x) = 0$ for $x < 0$) only at a finite distance downslope of the ridge when enough upslope topographic distance exists for the nonlocal contribution to substantially contribute to the sediment flux at a given point. Hence overall, the steady state hillslope equilibrium profile is parabolic near the ridgetop and becomes, shortly after, a power law profile with an exponent α (given by equation (16)). Further, we note that the steady state solution to the fractional diffusion equation predicts power law relationships of local gradient and curvature with downslope distance given by

$$-\nabla h \sim x^{\alpha-1} \tag{17}$$

$$\nabla^2 h \sim x^{\alpha-2} \tag{18}$$

That is, the fractional flux law predicts that curvature downslope of the ridge is not constant but decreases with downslope distance in a manner dictated by the exponent α (such a decrease has been documented, for example, in field observations in the work of *Roering et al.* [1999]). For $\alpha = 2$ the nonlocal transport model reproduces the linear profile in gradient and constant curvature with downslope distance, as expected for linear diffusive transport, while values of α between 1 and 2 give the flexibility of reproducing a suite of observed hillslope profiles. In section 4, we analyze field data from several real hillslopes and show that they are consistent with the nonlocal hypothesis of sediment flux.

4. Observed Hillslope Profiles Interpreted Within the Nonlocal Transport Theory

[14] The one-dimensional nonlocal theory presented here applies to hillslope profiles in which transport is assumed to be only along that profile, i.e., a one-dimensional approximation. Hillslopes, however, typically have significant contour (planform) curvature (i.e., ridges and hollows) and at steady state such curvature can accommodate the increasing soil production that must be carried downslope such that a single profile along the hillslope can be straight even in the case of linear flux-dependent transport and spatially constant erosion rates. Only a few detailed studies

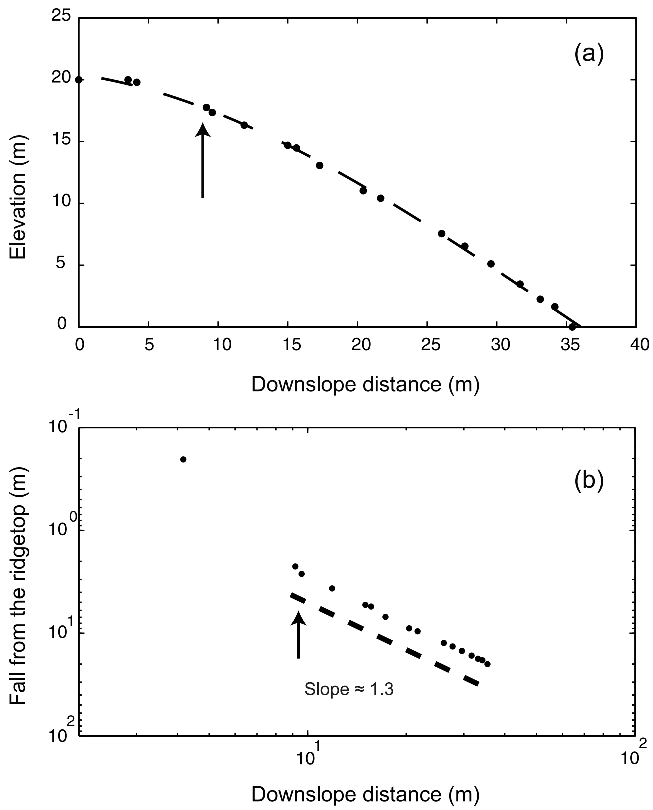

Figure 3. (a) A hillslope profile in the Oregon Coast Range. Solid circles represent the observed data points (reproduced from *Roering et al.* [1999]) and the dashed line indicates a spline fit to the observations. (b) Log-log plot of the fall from the hilltop versus horizontal distance for the above profile. Notice the power law profile with exponent 1.3 starting at a distance of 9 m from the ridgetop (arrow) consistent with a nonlocal transport law with parameter α = 1.3.

of hillslope form and process have been reported on hillslopes without significant planform curvature. Here we reexamine three well-known study sites (one clearly lacking planform curvature) and interpret them within the proposed nonlocal flux theory.

[15] *Roering et al.* [1999] motivate their work on nonlinear flux laws by reporting hillslope profiles in the Oregon Coast Range that clearly deviate from parabolic shape or constant curvature. Their study site experiences large scale disturbances due to massive tree throw mounds [*Heimsath et al.*, 1999], mammal burrowing and periodic fire [*Roering and Gerber*, 2005] and there is evidence for approximate steady state with considerable local variation over time scales of hillslope soil adjustment and development [*Roering, et al.*, 1999; *Heimsath et al.*, 2001; *Reneau and Dietrich*, 1991]. One of their profiles is shown in Figure 3a and the log-log plot of elevation fall versus horizontal distance (Figure 3b) suggests a slope of 1.3 for distances beyond 10 m downslope of the ridgetop and a slope of 2 close to the ridge (only 3 points are shown in Figure 3b at distance 0 to 10 m, but the slope of 2 is supported by more points obtained from the interpolated profile shown by the dashed line in Figure 3a). This profile is consistent with the nonlocal flux hypothesis and suggests that the nonlocal transport model proposed herein might be an alternative to the nonlinear model of *Roering et al.* [1999]. The conceptual bases of these two models are fundamentally different as they hypothesize different mechanisms of erosion and transport. This profile will be further analyzed in section 5.

[16] In their seminal paper on the geomorphology and forest ecology of the Shenandoah River area of Virginia, *Hack and Goodlett* [1960] report the result of plotting fall against distance for both their intensely surveyed study site and for a broad survey of 27 hillslopes in the Appalachians in Maryland and Virginia. They propose that the many regularities of the landforms and soils in the studied regions suggest steady state landscape adjustment. Ignoring the data points close to the divide, they report log-log linear profiles with a slope of 1.23 for the survey site and values ranging from close to 1 up to 1.7 for Maryland and Virginia. It is not clear how the broad survey data were collected (in the field versus from available topographic maps), nor whether they avoided slopes with planform curvature, but it is worth noting that the profiles do not include data points near the divide. They conclude that steeper hillslopes are generally straight (α values close to 1) and gentle ones more curved (α values closer to 2). Within our theory, this would suggest nonlocal transport on steeper hillslopes and local transport (linear diffusion) on gentle slopes. *Hack and Goodlett* [1960] describe soil transport as being driven by "growing roots, burrowing animals, falling raindrops, frost, tree blowdowns and the like" (p. 58). These processes would create a wide range of transport distances for a given slope. Specific localities and erosion rates for the hillslope profiles are not reported, so we must consider this suggestion as only a possibility, not an established condition.

[17] *McKean et al.* [1993] selected a hillslope transect with minimal planform curvature in the grasslands east of San Francisco, CA underlain by marine shales and documented soil transport rates using ^{10}Be concentrations in the clay-rich soils (Figure 4). From analysis of three soil pits within the first 35 m of hillslope length (from the ridge) they found evidence for a linear flux law and quantified the diffusive rate constant K (i.e., equation (1)). The soil transport occurs by seasonal creep of the high-plastic clay with biogenic transport being of some importance near the divide. Soil thickness varies inversely with curvature, consistent with a balance between soil production and linear transport [*Yoo et al.*, 2005, 2006]. The thickness is about 40 cm near the ridge and then increases downslope. Boundary conditions (channel incision rate and history) strongly influence hillslope profiles and at this study site the hillslope terminates in a broad, aggraded valley, which has led to a break in slope at the base of the hillslope and progressive thickening of soil toward the valley axis [*Yoo et al.*, 2005]. Both *Yoo et al.* [2005] and *McKean et al.* [1993] suggest that the upper smoothly convex hillslope could be at approximate steady state erosion, that is, the effect of stabilization of the lower boundary has not reached to the divide.

[18] We used the survey data collected by *McKean et al.* [1993] to construct the longitudinal profile reported in

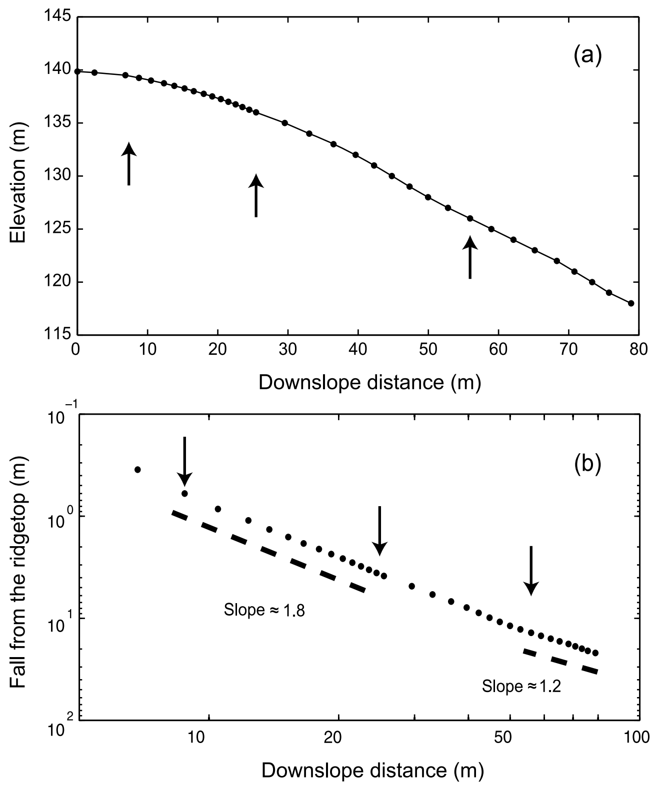

Figure 4. (a) Longitudinal profile of a hillslope reproduced from the survey data collected by *McKean et al.* [1993]. (b) Log-log plot of the fall from the hilltop versus horizontal distance. Notice the power law regime with exponent 1.8 starting at approximately 8 m from the ridgetop until 25 m downslope. This profile is consistent with a nonlocal flux hypothesis with exponent $\alpha = 1.8$. The abrupt transition to a slope of 1.2 on the lower portion of the hillslope is indicative that this part is still experiencing changes from net erosion to progressive soil accumulation.

Figure 4a. By plotting on a log-log scale the elevation fall versus horizontal distance from the ridge (Figure 4b) we observe a slope of ≈1.8 from a distance of 8 m from the ridgetop up to approximately 25 m downslope; in the first 8 m from the ridgetop one would expect a parabolic profile (slope of 2). The hillslope rapidly flattens upslope from 8 m and the available survey data do not provide adequate constraint on the profile shape. The gentle hillslope gradient and high clay content (which favors creep) and the dry, grassy, relatively low biota mantle on the convex hilltop all would favor an almost local transport, and the slope value of 1.8 extending for the first 25 m is consistent with this expectation. Downslope of 50 m to the lowest portion of the hillslope surveyed the slope of the power law plot of elevation against distance is ≈1.2. This transition is not consistent with the nonlocal flux law of $\alpha = 1.8$ discussed above; rather the bottom part of the hillslope is interpreted as experiencing a change from net erosion to progressive soil accumulation (due to lower boundary conditions) and field observations support this interpretation. This example illustrates that the nonlocal flux theory can also be used as a diagnostic tool for inferring process from form and further motivate data collection to test alternative hypotheses.

5. Nonlocal Versus Nonlinear Flux: Same Behavior for Different Reasons

5.1. Nonlinear Transport Model as an Emulator of Superdiffusivity

[19] Deviation from purely diffusive behavior in many hillslopes has prompted the development of more complex transport laws which have a nonlinear dependence on topographic gradient. A review of several of these laws can be found in the work of *Dietrich et al.* [2003]. For example, for soil mantled hillslopes, *Roering et al.* [1999] proposed the following flux equation [see also *Andrews and Buckman*, 1987; *Howard*, 1994]:

$$q_s' = \frac{K\nabla h}{1 - (|\nabla h|/S_c)^2} \tag{19}$$

where q_s' is the sediment flux calculated at a point via the nonlinear flux law, K is the diffusivity coefficient, and S_c is called the "critical gradient." It is noted that the above equation imitates a superdiffusive behavior, that is, close to linear diffusion at low slopes and accelerated diffusion at high slopes. Although this can be directly seen from (19), it is interesting to see it from a different perspective. By substituting (19) in (2) and performing a Taylor series expansion we obtain

$$\frac{\partial h}{\partial t} = K\nabla^2 h + K\frac{\nabla^2 h}{S_c^2}(|\nabla h|)^2 + \dots \tag{20}$$

The second term in the RHS of (20) shows that the nonlinear transport law of (19) captures the superdiffusive behavior at high slopes by enhancing the regular diffusion with the addition of a term that has an explicit nonlinear dependence on gradient. The gradient in the above equation is "local." We propose that such superdiffusive behavior in steep hillslopes can be addressed using nonlocal transport laws, which are linear (i.e., they involve only linear combinations of local gradients) but take into account that disturbances contributing to sediment flux at a point of interest have an origin far upslope of that point. It is interesting to note that the proposed nonlocal flux law gives rise to a nonlinear dependence of sediment flux on the local gradient at any point (this will be presented in section 5.2) but for reasons different than the explicit quadratic dependence of flux on local gradient as in equation (20).

5.2. Nonlocality Gives Rise to a Nonlinear Dependence of Flux on Local Gradient

[20] We use the *Roering et al.* [1999] hillslope profile from the Oregon Coast Range to illustrate the computation of the sediment flux from the nonlocal transport model of (8) and compare it to those of the linear (1) and nonlinear (19) models. In order to have a continuous set of elevation data points over the domain of interest, the observations were interpolated using a spline as shown in Figure 3a with dashed lines.

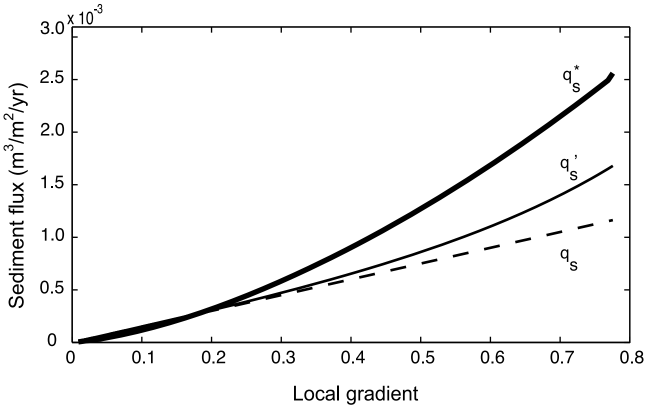

Figure 5. Comparison of the three flux laws. The dashed line shows the sediment flux predicted by linear, local flux law (equation (1)). The thick line shows the sediment flux predicted by the linear, nonlocal law (equation (8)), and the thin line shows the sediment flux predicted by local, nonlinear law (equation (19)). The parameters for q'_s are chosen to be $K = 0.0015\text{m}^2/\text{yr}$ and $S_c = 1.4$ (from *Roering et al.* [1999]). The parameters for the calculation of q^*_s are chosen to be $\alpha = 1.5$ and $K^* = 0.0007\text{m}^{1.5}/\text{yr}$.

[21] The computation of the fractional flux was performed on a discrete grid of size Δx by the Grünwald-Letnikov discrete approximation of the fractional integral operator given as [*Grünwald*, 1867; *Podlubny*, 1999]

$$\nabla^{\alpha-1}h(x) = I_x^{2-\alpha}\{\nabla h(x)\} = \lim_{\Delta x \to 0} (\Delta x)^{2-\alpha} \sum_{k=0}^{x/\Delta x} \frac{\Gamma(2-\alpha+k)}{k!\Gamma(2-\alpha)} \nabla h(x - k\Delta x) \quad (21)$$

It is noted that writing the fractional flux as a fractional integral of the local slopes (first equality in the above equation) is enabled by use of the Caputo definition of the fractional derivative (equation (11b)).

[22] The parameters chosen for the three flux laws (linear, nonlinear and nonlocal) are $K = 0.0015\text{m}^2/\text{yr}$, $S_c = 1.4$, $\alpha = 1.5$ and $K^* = 0.0007\text{m}^\alpha/\text{yr}$. The model parameters for the nonlinear flux law are chosen from the ones calibrated for Oregon Coast Range in the work of *Roering et al.* [1999]. For the nonlocal flux law, α is set to 1.5 and K^* is chosen such that all the three flux laws show a similar increase in sediment flux with slope at lower gradients. This is done in order to study the effect of the three flux laws at the higher gradients. Figure 5 shows the sediment flux computed using the three different flux laws. The sediment flux computed from the nonlocal transport law (8) shows a similar behavior as the nonlinear, local transport law (equation (19)), with enhanced diffusion at higher gradients. Hence, a nonlinear relationship between sediment flux and local gradient can also arise from a nonlocal, linear flux model. It is emphasized that in a real hillslope, the parameters K for the nonlinear model and K^* for the nonlocal model are obtained via calibration; the unfamiliar units of K^* (L^α/T) are not an issue and simply reflect that the quantity $(K^*t)^{1/\alpha}$ maintains the units of length (length scale of diffusion) in analogy to the quantity $(Kt)^{1/2}$ for standard diffusion [see, e.g., *Benson*, 1998].

5.3. Nonlocality and Upslope "Region of Influence"

[23] The nonlocal transport law differs from any local transport law (linear or nonlinear) in that in the former, the sediment flux contribution to a given point on the hillslope is computed from a weighted average of the topographic gradients upslope of that point. Therefore, unlike the local transport laws, the nonlocal transport law has a "memory" of the upslope topography. Although the power law kernel $g(l)$ of the nonlocal integral flux (equation (7)) implies lack of characteristic scale over which the averaged gradient is computed, we take the liberty below to introduce a cutoff scale in order to illustrate this upslope influence effect. Specifically, we introduce a physically tangible measure of nonlocality for the computation of sediment flux by defining an *influence length*, L_α, as the distance upslope from a given point, beyond which the contribution of the sediment flux is less than 10% of the total; that is, L_α is defined by the equation

$$K^* \int_0^{L_\alpha} g(l)\nabla h(x-l)dl \approx 0.9q_s^*(x), 1 < \alpha < 2 \quad (22)$$

where $g(l) \sim l^{1-\alpha}$ are the weights given to the gradients uplsope and q^*_s is the nonlocal flux calculated by (8). The cutoff of 10% is chosen here arbitrarily to illustrate the behavior of nonlocal flux and it can be chosen to be lower or higher depending on the problem at hand.

[24] The influence length was calculated for the *Roering et al.* [1999] profile from equation (22) for three different values of α and is shown in Figure 6. The degree of nonlocality increases with a decrease in α; that is, the closer the value of α is to 1.0 the more nonlocal the transport is compared to a value of α closer to 2. As expected, a higher degree of nonlocality results in a larger value of L_α as seen in Figure 6. For $\alpha = 2$, equation (22) is not applicable for computation of the influence length. In this case, the step

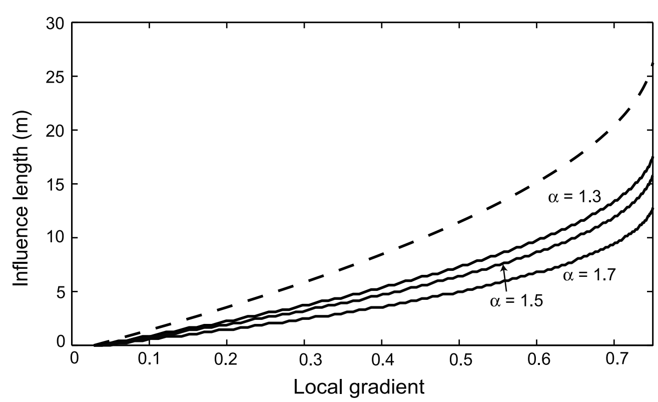

Figure 6. Plot showing the upslope influence length L_α (see text for definition) as a function of local gradient and degree of nonlocality for the hillslope of *Roering et al.* [1999]. The dashed line indicates the distance to the ridgetop, in other words, the maximum available distance to take part in the transport.

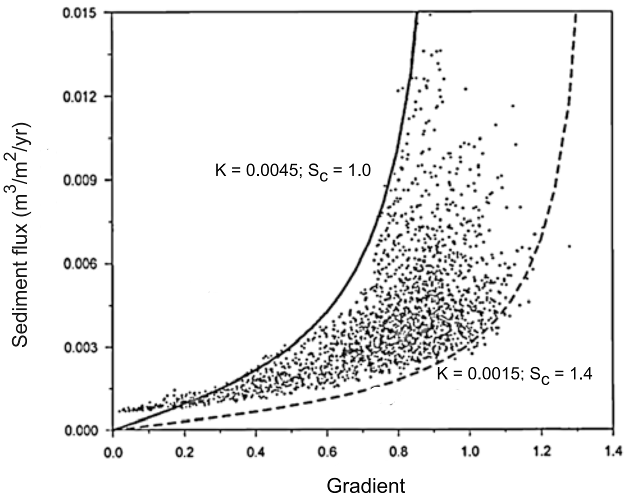

Figure 7. Reproduced from *Roering et al.* [1999] to illustrate the large natural variability of calculated sediment flux (dots) even in a small hillslope (MR1 basin in Oregon Coast Range; sediment flux calculated using equation (23)) and the wide range of fitted parameters K (m^2/yr) and S_c that would be needed to reproduce the observed variability under the assumption of a nonlinear local transport law.

lengths have a thin-tailed distribution whose characteristic scale (standard deviation) can be used to define the influence length.

6. Nonlocality Naturally Reproduces Spatial Variability of Sediment Flux

[25] In section 5, all the flux laws were discussed in the context of a single hillslope profile. However, even in a small hillslope, there exists considerable variability in the form of hillslope profiles which results in a considerable variability in the observed sediment flux. This flux variability was documented by *Roering et al.* [1999] for the MR1 basin of Oregon Coast Range. They computed the sediment flux using

$$q_s = U \frac{\rho_r}{\rho_s} \frac{a}{b} \tag{23}$$

where U is the constant rock uplift rate, ρ_r and ρ_s are bulk densities of rock and sediment, respectively, and a/b is the drainage area per unit contour length, and compared it against the flux computed from their nonlinear transport model. Figure 7 (reproduced from *Roering et al.* [1999]) shows the spread of the computed sediment flux as a function of gradient. Notice that for a given gradient, say for a gradient of 0.8 there is an order of magnitude variability in the computed flux. To describe this variability with the nonlinear law, equation (19), the calibrated parameters of the model had to vary considerably: K = 0.0015m^2/yr to 0.0045m^2/yr and S_c = 1.0 to 1.4 as reported by *Roering et al.* [1999]. We note that S_c is a calibration parameter which was attached a physical meaning of a critical slope and was related to the angle of repose in the work of *Roering et al.* [1999]; later in the work of *Roering and Gerber* [2005] it

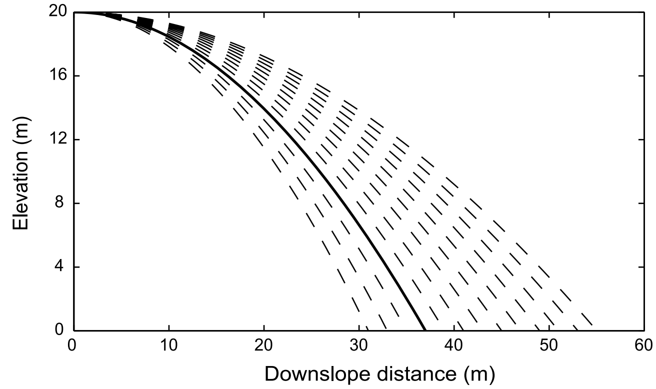

Figure 8. Plot showing the suite of generated hillslope profiles to imitate the natural variability of profiles (flow paths perpendicular to contour lines) in a zero-order basin. The thick line indicates the profile reproduced from *Roering et al.* [1999].

was proposed that K increased and S_c decreased in response to forest fire.

[26] Here we pose the hypothesis that a nonlocal transport model can capture the observed variability of sediment flux within a given hillslope by a single or very narrow range of parameters, unlike any local transport law. To test the hypothesis, we generated a set of hillslope profiles using different cubic polynomials (see Figure 8) to imitate the natural variability of hillslope profiles within a small basin. Along those profiles the sediment flux was computed using the nonlocal, linear flux model (equation (8)) and local, nonlinear flux model (equation (19)). Figure 9 shows the computed sediment flux as a function of the local gradient.

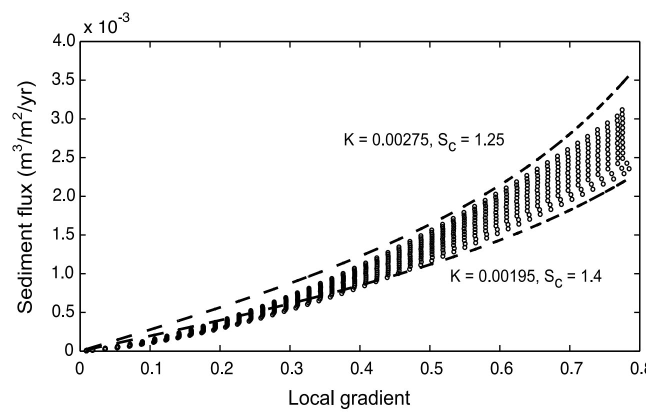

Figure 9. Sediment flux computed on the suite of hillslope profiles (shown in Figure 8) using the linear, nonlocal transport law (equation (8)) with parameters α = 1.5 and K^* = 0.0007m^α/yr (open circles). Note that while these parameters are kept constant, a large variability of the sediment flux is produced due to the variability in the ensemble of profiles. In order to reproduce this variability with the nonlinear transport law (equation (19)), the range of fitted parameters required (concentrating on the higher gradients where the nonlinear transport law is more pertinent; see also Figure 7) is K = 0.00195m^2/yr and S_c = 1.4 and K = 0.00275m^2/yr and S_c = 1.25 (broken lines).

The nonlocal transport law with a single set of parameters K^* and α produces a variability of sediment flux for a given gradient comparable to that observed in real hillslopes (Figure 7). However, the local transport law cannot reproduce this variability with a single set of parameters K and S_c but requires a considerable range of parameter values as indicated by the envelope curves in Figure 9. This is simply because two points with the same local slope would result in the same flux from any local transport law but different fluxes from a nonlocal law, due to different upslope topography. Having the need for such a wide range of parameters to reproduce the sediment flux variability in a small hillslope makes physical interpretation of those parameters difficult. Apart from the upslope hillslope profile variability considered here, there are other factors contributing to the sediment flux/local gradient variability, such as, for example, the dependence of K on soil depth [e.g., *Roering*, 2008].

7. Probability Distribution of Particle Displacement and Fractional Transport

[27] Sediment transport on hillslopes can be thought of as disturbance driven, in which soil is mobilized en masse or as individual particles. A single disturbance event may move the mobilized sediment a considerable distance (e.g., raveling after a fire). Disturbed piles of sediment (e.g., tree throw mounds) will create sustained local areas of elevated flux and increased downslope delivery. For simplicity we can think of event-based transport as a kind of "hopping" process, where the sediment moves downslope in a series of steps resulting from local disturbances. Here a single hop can be thought of as the distance covered by a grain of sediment from where disturbance has displaced it into an active flux state to where it comes to transient rest (until next disturbance). It can also be thought of as a package of sediment made significantly more active due to local mounding and exposure, say during a tree throw, which results in rapid flux compared to what would happen under mean slope conditions. As discussed in the introduction, many processes generate slope-dependent transport and operate over a wide range of distances. These processes may result in a heavy-tailed PDF of the sediment "hops" or displacement distance [see also *Tucker and Bradley*, 2010], which means that there is a relatively small but significant possibility that sediment grains will move a great distance downslope in a single hop. In other words, these distances do not have a characteristic length scale and may assume values comparable to the size of the hillslope itself.

[28] If the PDF of hopping distances were thin tailed, e.g., Gaussian or exponential with an e-folding distance small relative to the size of the hillslope, then the continuum equation describing the evolution of the hillslope would be the diffusion equation [*Feller*, 1971; *Schumer et al.*, 2009]. However, if the probability distribution of hopping distances is broad tailed as argued above, then a faster than linear diffusion is expected. It turns out that, since a sum of broad-tailed pdfs results in an α stable distribution for the hopping process [*Feller*, 1971], then the governing equation of elevation change consistent with this distribution is the fractional diffusion equation (9) [*Meerschaert et al.*, 1999, 2001; *Schumer et al.*, 2009]. That is, the corresponding macroscopic process of sediment transport can be described using a modified diffusion equation where the ∇^2 operator is replaced with a nonlocal operator ∇^α. The degree of nonlocality is governed by the order of differentiation, α. The lower the value of α, the greater is the degree of nonlocality. This is a manifestation of the fact that an α stable PDF has a heavier tail for lower values of α.

8. Locality and Scale Dependence of Computed Flux

[29] In this section we discuss some preliminary ideas related to the potential of nonlocal transport laws to circumvent the problem of scale dependence of sediment flux computations. We start with the classical divergence theorem and elementary control volume which is of little use when there is no characteristic scale in particle displacement distances. Then, we allude to the fact that local transport laws suffer from scale dependencies which would require closures [see, e.g., *Passalacqua et al.*, 2006] and which can be naturally taken care of by the nonlocal transport laws.

[30] The advection-dispersion equation (ADE) is based on the classical definition of divergence of a vector field. The divergence is defined as the ratio of total flux through a closed surface to the volume enclosed by the surface when the volume shrinks to zero [e.g., *Schey*, 1992] (see also *Benson* [1998] for an exposition relevant to subsurface transport):

$$\nabla \cdot q_s = \lim_{V \to 0} \frac{1}{V} \iint_S q_s \cdot \eta dS \tag{24}$$

where q_s is a vector field, V is an arbitrary volume enclosed by surface S, and η is a unit normal vector. Implicit in this equation is that the limit of the integral exists; that is, the vector q_s exists and is smooth as $V \to 0$.

[31] The classical notion of divergence maintains that, as an arbitrary control volume V shrinks to zero, the ratio of total surface flux to volume must converge to a single value. However, when a heavy-tailed distribution of displacement lengths exists, this notion of convergence is challenged. In fact, due to the lack of a characteristic scale of the displacement distances, no convergence is guaranteed when the size of the control volume changes. As a result, the classical diffusion equation is no longer self-contained with a closed form solution at all scales. To adopt the classical theory, the best approximation that can be done is to assume that the total flux to volume ratio can be assumed piecewise constant within small ranges of scales, allowing one to talk about an "effective" scale-dependent dispersion coefficient [see, e.g., *Benson*, 1998]. Several techniques have been proposed in the subsurface transport literature to tackle the problem of scale-dependent dispersivity. These vary from small perturbation approaches and effective parameterizations [e.g., *Gelhar and Axness*, 1983; *Dagan*, 1997], to power law dependence of D on scale [e.g., *Su*, 1995], to volume statistical averaging [e.g., *Cushman*, 1991, 1997] and to fractional advection-dispersion equations (fADE) [e.g., *Benson*, 1998; *Benson et al.*, 2000b; *Baeumer et al.*, 2001; *Schumer et al.*, 2001, 2009].

[32] Any sediment transport law that directly involves a "local" gradient or curvature in the computation of flux, will

be scale dependent as gradients and curvatures depend on the scale at which they are computed [see, e.g., *Lashermes et al.*, 2007]. For example, this was demonstrated by *Passalacqua et al.* [2006] using a local nonlinear flux law (a Langevin model which has square dependence on local slope). In that study, the development of a closure term, akin to the Large Eddy Simulation (LES) turbulence closures, was proposed to handle this scale dependence and the closure term was shown to have a power law dependence on scale (grid size). The proposed nonlocal fractional diffusive model has in principle the ability to remove this scale dependency as it is free of any "representative" or "control volume" concept and the power law integration of local gradients (see equation (21)) eliminates the need for the aforementioned power law closure [see, e.g., *Foufoula-Georgiou et al.*, 2008]. This issue requires further study.

9. Discussion and Conclusions

[33] Most geomorphic transport laws proposed to date are local in character; that is, they express the sediment flux or erosion at a point as a function of the elevation gradient, contributing drainage area, or other geomorphic quantities at that point only. For the case of soil-mantled landscapes, it is reasonable to propose that disturbance processes inducing transport have widely varying transport distances and this gives rise to a nonlocality of sediment transport, as proposed here. As summarized below, we see several advantages to the nonlocal transport law.

[34] 1. The proposed nonlocal transport model with boundary conditions of zero slope at the ridgetop and constant elevation at the ridge bottom predicts a steady state profile which is parabolic very close to the ridgetop and changes, after a short distance downslope to a power law with exponent equal to the parameter α (order of differentiation) in the fractional transport law. This prediction is supported by data in three study sites and provides useful insight for one of the sites which may still experience transition from net erosion to soil accumulation.

[35] 2. The nonlocal linear model gives rise to a nonlinear relationship between the sediment flux at a point and the local slope. Hence, nonlocality of sediment flux is an alternative hypothesis that can explain the observed hillslope profiles and the nonlinear flux dependence on slope.

[36] 3. In a practical implementation of a local sediment flux law (linear or nonlinear), the "local" slope is always assigned a "scale" over which some smoothing or averaging is done, without however a theory as to how to select this scale. The nonlocal flux law is scale free (it lacks a characteristic scale of upstream particle displacement distance); rather it uses a "power law weighted average slope" stating that upslope hillslope gradients matter to local flux, but with diminishing influence as a function of upslope distance.

[37] 4. The proposed nonlocal model produces significant variability of sediment flux for a given local slope, as it explicitly takes into account variations in upslope topography. In this case, transport parameters, such as K^*, can remain constant, and retain, perhaps, a stronger physical meaning while reproducing the variability observed in real hillslopes.

[38] 5. The nonlocal model has the potential to eliminate scale dependency. The usefulness of nonlocal fractional models to address issues of scale dependence in subsurface transport (e.g., scale-dependent dispersivity in porous media with multiple scales of heterogeneity) has been amply demonstrated and needs to be explored for similar problems on the Earth's surface.

[39] We consider this paper as the beginning of a dialogue on concepts of nonlocality and collective behavior as they relate to transport on the earth's surface. Important questions arise as to how these concepts can most concisely be expressed in or incorporated into new classes of geomorphic transport laws and also how nonlocality can be directly verified from observations. Together with several other papers in this issue, a new direction of thinking emerges which shows promise for better understanding of cause and effect in landscape processes and landscape evolution models.

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